# Norm of Bdd. (Continuous) Linear Transformation

**Defn.** Let $N$ & $N'$ be NLS and let $T$ be a bdd. L.T. on $N$ into $N'$.

We define the norm of $T$ by →

$$\|T\| = \sup\{\|T(x)\| \mid x \in N, \|x\| \leq 1\}$$

This is also called operator Norm.

**Note** → Here we consider those elements in $N$ whose norm is $\leq 1$ i.e. $\|x\| \leq 1$

Then we consider images of all these elements. i.e. $T(x)$.

We find supremum of norm of all these images i.e. $\sup\{\|T(x)\| \mid \|x\| \leq 1\}$

That supremum will be operator Norm.

**Thm** → Let $N$ and $N'$ be NLS and let $T$ be a bdd L.T. of $N$ into $N'$. Put —

$$a = \sup\{\|T(x)\| \mid x \in N, \|x\| = 1\}$$

$$b = \sup\left\{\frac{1}{\|x\|}\|T(x)\| \mid x \in N, x \neq 0\right\}$$

$$c = \inf\{k \mid k \geq 0, \|T(x)\| \leq k\|x\|, \forall x \in N\}$$

Then $\|T\| = a = b = c$

and $\|T(x)\| \leq \|T\|\,\|x\| \quad \forall x \in N$

Proof → We know that —

$$\|T\| = \sup\{\|T(x)\| \mid x \in N, \|x\| \le 1\} \rightarrow (1)$$

, by def$^n$ of norm

Now →

$$\|T(x)\| \le c\|x\|, \quad \forall x \in N$$

→ by def$^n$ of c

If $\|x\| \le 1$ then $\|T(x)\| \le c \quad \forall x \in N$

Since $\|T(x)\| \le c \quad \forall x \in N$

$$\therefore \sup\{\|T(x)\| \mid x \in N, \|x\| \le 1\} \le c$$

$$\therefore \|T\| \le c, \quad \text{from eq}^n\ (1) \rightarrow (2)$$

By def$^n$ of b & c, it is evident that →

$$c \le b. \rightarrow (3)$$

Now, if $x \ne 0$ then

$$\frac{1}{\|x\|}\|T(x)\| = \left\|\frac{1}{\|x\|}T(x)\right\| = \left\|T\left(\frac{x}{\|x\|}\right)\right\|$$

Also, $\left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1$

∴ From def$^n$ of a & b, we get →

$$b \le a. \rightarrow (4)$$

But $a \le \|T\|$, ∴ we get

$$\underbrace{\|T\| \le c}_{\text{from }(2)} \underbrace{\le b}_{\text{from}(3)} \underbrace{\le a}_{\text{from}(4)} \le \|T\| \Rightarrow \|T\| = a = b = c$$

From def$^n$ of $b$, we get →

$$\frac{1}{\|x\|}\|T(x)\| \leq b = \|T\|$$

$$\therefore \|T(x)\| \leq \|T\|\,\|x\|$$

(Proved)

Note → The set of all bdd (continuous) L.T. from $N$ into $N'$ is represented by $B(N,N')$. $B(N,N')$ is a normed linear space w.r.t. pointwise linear operations —

$$(T+U)(x) = T(x) + U(x)$$

& $(\alpha T)(x) = \alpha T(x)$

The norm is defined by →

$$\|T\| = \sup\left\{ \|T(x)\| \,\middle|\, x \in N,\ \|x\| \leq 1 \right\}$$

If $N'$ is a Banach Space then $B(N,N')$ is also Banach Space.

Ex → Let $N, N'$ be NLS. Let $T$ be a L.T. from $N$ into $N'$. Then $T$ is continuous either at every point of $N$ or at no point of $N$.

Sol$^n$: Let $x_1, x_2 \in N$, $x_1 \neq x_2$

Let $T$ be continuous at $x_1$.

We shall show that $T$ is cont. at $x_2$ also.

Since $T$ is cont. at $x_1$,

$\therefore$ for any $\epsilon > 0$, $\exists$ $\delta > 0$ s.t.

$$\|T(x) - T(x_1)\| < \epsilon \quad \text{whenever } \|x - x_1\| < \delta \quad \rightarrow (1)$$

Now $\rightarrow$

$$\|x - x_2\| < \delta \Rightarrow \|x + x_1 - x_1 - x_2\| < \delta$$

$$\Rightarrow \|(x + x_1 - x_2) - x_1\| < \delta$$

$$\Rightarrow \|T(x + x_1 - x_2) - T(x_1)\| < \delta$$

, using eqn (1)

$$\Rightarrow \|\underbrace{T(x) + T(x_1) - T(x_2)}_{\because \, T \text{ is linear}} - T(x_1)\| < \delta$$

$$\Rightarrow \|T(x) - T(x_2)\| < \epsilon$$

We have shown that for any $\epsilon > 0$, $\exists$ $\delta > 0$ s.t. $\|T(x) - T(x_2)\| < \epsilon$ whenever $\|x - x_2\| < \delta$

$\therefore$ $T$ is cont. at $x_2$ also.

$\therefore$ $T$ is cont. at all points of $N$.

(Proved)

Ex $\rightarrow$ Let $N, N'$ be NLS over same field of scalars. Let $T$ be L.T. from $N$ into $N'$. Then $T$ is bdd iff it is continuous.

Soln $\rightarrow$ Let $T$ be bdd. We shall show that $T$ is continuous.

Since $T$ is bounded, $\therefore \exists$ some $M > 0$

s.t. $\|T(x)\| \leq M\|x\|$, $\forall x \in N$ $\rightarrow (1)$

Let $x \in N$ be arbitrary.

Choose $\delta = \frac{\epsilon}{M}$ for any $\epsilon > 0$.

Then $\forall y \in N$ s.t. $\|y - x\| < \delta$,

we have $\rightarrow$

$$\|T(y) - T(x)\| = \|T(y-x)\|$$

$$\leq M\|y-x\| \quad , \text{from}(1)$$

$$< M\delta = M.\frac{\epsilon}{M} = \epsilon$$

$\therefore \|T(y) - T(x)\| < \epsilon$

$\Rightarrow T$ is cont.

<u>Converse</u> Let $T$ be cont.

To show that $T$ is bdd

We shall prove it by contradiction.

Suppose $T$ is not bdd. i.e. $\exists$ no $K > 0$ s.t.

$$\|T(x)\| \leq K\|x\| \quad , \forall x \in N$$

Then $\exists$ a point $x_n \in N$, for each +ve integer $n$, such that

$$\|T(x_n)\| > n\|x_n\| \rightarrow (2)$$